January 25, 1883.

THE PRESIDENT in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Paper was read:—

I. "On certain Definite Integrals." No. 11. By W. H. L. Russell, F.R.S. Received January 16, 1883.

The method by which integral (227) was obtained may be thus extended.

Suppose it was required to obtain

$$\int dx f\left(\frac{a+bx+cx^2+ex^3+\ldots}{a'+b'x+cx^2+e'x^2+\ldots}\right).$$

Put
$$z=\frac{a+bx+cx^2+ex^3+\ldots}{a'+b'x+c'x^2+e'x^3+\ldots},$$

$$\theta(x)=a+bx+cx^2+ex^3+\ldots,$$

$$\phi(x)=a'+b'x+c'x^2+e'x^3+\ldots,$$

then $\theta(x)=z\phi(x)$, and if (α) be any root of the equation $\theta(x)=0$, and $\theta', \theta'', \ldots \phi', \phi'', \ldots$, are the values of $\theta'x$, $\theta''x$ when $x=\alpha$, then we find by differentiating $\theta(x)=z\phi(x)$ that

$$x=U_0+U_1+U_2\frac{z^2}{1.2}+U_3\frac{z^3}{1.2.3}+\ldots$$

when
$$U_0=\alpha_1, \quad U_1=\frac{\phi}{\theta'}, \quad U_2=\frac{2\phi\phi'}{\theta'^2}-\frac{\theta''\phi^2}{\theta'^3},$$

$$U_3=\frac{3\phi''\phi^2}{\theta'^3}-\frac{9\phi^2\phi'\theta''}{\theta'^4}+\frac{6\phi\phi'^2}{\theta'^3}-\frac{\theta'''\phi^3}{\theta'^4}+\frac{3\theta''\phi^3}{\theta'^5}.$$

Hence
$$\int dx f\left(\frac{a+bx+cx^2+ex^3+\ldots}{a'+b'x+c'x^2+e'x^3+\ldots}\right)$$

$$=\int dz\left(U_1+U_2z+U_3\cdot\frac{z^2}{1.2}+\ldots\right)f(z) \quad . \quad (234).$$

The theorem applies with great facility when $f(z)=\sqrt[n]{z}$. I of course

suppose that $f(z)$ can be expanded in terms of z by a converging series in other cases. The series $U_1+U_2z+\ldots$ would, I believe, be convergent in many cases, where θ' is large when compared with $\phi, \phi', \ldots$; but the subject requires further consideration.

If we put $x=0$, $$z=\frac{a}{a'}.$$

Hence we immediately deduce from the equation

$$x=U_0+U_1z+U_2\frac{z^2}{1\,.\,2}+\ldots \quad . \quad . \quad . \quad . \quad (235)$$

the theorem $$U_0+U_1\frac{a}{a'}+\frac{U_2}{1\,.\,2}\frac{a^2}{a'^2}+\frac{U_3}{1\,.\,2\,.\,3}\frac{a^3}{a'^3}+\ldots=0.$$

Since $$\int_0^1\!\!\int dx x^{n-1}=\frac{1}{n(n+1)},$$

$$\int_0^1\!\!\int dx(1+x)x^{n-1}=\frac{1\,.\,2}{n(n+1)(n+2)},$$

$$\int_0^1\!\!\int dx(1-x)x^{n-1}=\frac{1\,.\,2\,.\,3}{n(n+1)(n+2)(n+3)},$$

and, therefore, as before,

$$\int_0^1\!\!\int dx x^{n-1}\phi(1-x)$$

$$=\frac{A_0}{n(n+1)}+\frac{A_1\,.\,1\,.\,2}{n(n+1)(n+2)}+\frac{A_2\,.\,1\,.\,2\,.\,3}{n(n+1)(n+2)(n+3)}+\ldots \quad . \quad (236)$$

with precisely similar formulæ for $\int_0^1\!\!\int\!\!\int dx^3x^{n-1}\phi(1-x)$ and other like integrals.

We also have

$$\int_0^1\frac{\phi(1-x^2)dx x^{2n}}{\sqrt{1-x^2}}$$

$$=\left\{A_0+\frac{A_1}{2n+2}+\frac{A_2\,.\,1\,.\,3}{(2n+2)(2n+4)}+,\ \&c.\right\}\int_0^1\frac{dx x^{2n}}{\sqrt{1-x^2}} \quad . \quad (237),$$

$$\int_0^\omega\phi\left(\frac{x^2}{1+x^2}\right)\frac{dx}{(1+x^2)^n}$$

$$=\left\{A_0+\frac{A_1}{2n}+\frac{A_2\,.\,1\,.\,3}{2n(2n+2)}+,\ \&c.\right\}\int_0^\omega\frac{dx}{(1+x^2)^n} \quad . \quad . \quad (238).$$

It is easily seen that these investigations extend to a great multi-

tude of what are usually called binomial integrals. It will be seen that these formulæ include a vast number of such integrals as

$$\int \frac{dx \,.\, x^{2n}}{\sqrt{a^2-x^2}\sqrt{1-x^2}} \quad . \quad . \quad (239). \qquad \int \frac{dx \,.\, x^{2n}}{\sqrt[3]{a^2-x^2}\sqrt{1-x^2}} \quad . \quad . \quad (240).$$

$$\int \frac{dx \,.\, x^{2n}}{\sqrt{a+bx^2+cx^4+\ldots+ex^{2n}}\sqrt{1-x^2}} \quad . \quad . \quad . \quad . \quad . \quad (241).$$

$$\int \frac{dx \,.\, x^{2n}}{\sqrt[4]{a+bx^2+cx^4+\ldots+ex^{2n}}\sqrt{1-x^2}} \quad . \quad . \quad . \quad . \quad (242).$$

If we have an integral of the form

$$\int d\theta \epsilon^{\mathrm{P}x}\mathrm{Q} = \epsilon^{ax}b,$$

where P and Q are functions of θ, we have by applying the symbol $\phi\frac{d}{dx}$ to the integral

$$\int d\theta\phi(\mathrm{P})\epsilon^{\mathrm{P}x}\mathrm{Q} = \phi(a)a^x b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (243).$$

As an example of this

$$\int_0^{\frac{\pi}{2}} d\theta\{\phi(\cos^3\theta\epsilon^{3i\theta})\epsilon^{x\cos^3\theta\epsilon^{3i\theta}} + (\phi\cos^3\theta\epsilon^{-3i\theta})\epsilon^{x\cos^3\theta\epsilon^{-3i\theta}}\} = \pi\phi(\tfrac{1}{8})\epsilon^{\frac{x}{8}} \quad (244),$$

from whence

$$\int_0^{\frac{\pi}{2}} d\theta\epsilon^{x\cos^3\theta\cos_3\theta} \frac{\cos(x\cos^3\theta\sin 3\theta) + \mu\cos^3\theta\cos(x\cos^3\theta\sin 3\theta - 3\theta)}{1+2\mu\cos^3\theta\cos 3\theta + \mu^2\cos^6\theta}$$

$$= \frac{4\pi}{\mu+8}\epsilon^{\frac{x}{8}} \quad . \quad . \quad . \quad (245).$$

In the same way we may deduce general formulæ from integrals (86), (87), (116), (118), (122), (129), (130), of the present series.

The following integrals were obtained by the use of reciprocal functions:—

$$\int_0^{\pi} d\theta \cos^r\theta\epsilon^{a\cos\theta}\cos(a\sin\theta + r\theta) = \frac{\pi}{2^r} \quad . \quad . \quad . \quad . \quad (246),$$

$$\int_0^{\pi} d\theta\cos^r\theta \,.\, \frac{\cos^r\theta - \alpha\cos(r-1)\theta}{1-2\alpha\cos\theta+\alpha^2} = \frac{\pi}{2^r} \quad . \quad . \quad . \quad . \quad (247),$$

$$\int_0^{\pi} \frac{d\theta}{(1-2\alpha\cos r\theta+\alpha^2)(1-2\beta\cos s\theta+\beta^2)} = \frac{\pi(1+\alpha^s\beta^r)}{(1-\alpha^2)(1-\beta^2)(1-\alpha^3\beta^r)}$$

$$. \quad . \quad . \quad . \quad . \quad . \quad (248),$$

where r and s are prime numbers.

$$\int_0^{\pi} d\theta\{\epsilon^{a\epsilon^{\theta i}}\phi(\cos\theta\epsilon^{\theta i}) + \epsilon^{a\epsilon^{-\theta i}}\phi(\cos\theta\epsilon^{-\theta i})\} = 2\pi\phi(\tfrac{1}{2}) \quad . \quad . \quad (249).$$

This last formula is derived from (246). I observe that the fundamental idea by which these integrals are obtained is given by the equation

$$\int_0^{\pi}(A_0+A_1\cos\theta+\ldots A_r\cos r\theta)(B_r\cos r\theta+B_{r+1}\cos(r+1)\theta+\ldots)=\frac{\pi}{2}A_rB_r.$$

This method may be much extended.

II. "Internal Reflexions in the Eye." By H. FRANK NEWALL, B.A. Communicated by Dr. M. FOSTER, Sec. R.S. Received January 18, 1883.

1. The observation I have to record first came under my notice three or four years ago. Often when working at night by the light of a candle, in a room otherwise dark, my attention was caught by a very faint light some way out of the line of direct vision. This seemed to defy nearer inspection; for the instant I turned my eyes towards it, it was gone, thus showing that there was no objective cause, but that the light was due to some internal reflexion in the eye. Later, however, I found that by keeping the eye fixed and moving the candle, the faint light could be observed at leisure, though, as far as I could then make out, never in the line of direct vision. (See however below, § 32.)

2. The best conditions soon became apparent, and I have applied two methods in later investigations: (i) One in which the eye is fixed on a spot on a dark or uniform ground whilst the candle is moved to and fro out of the line of direct vision. (ii) One in which the candle or source of light is kept fixed, whilst the eye follows the regular movement of some point, such as the end of a pencil moved by the hand.

3. The first of these methods showed that the ghost, as I may call the faint light, moved roughly speaking in a line drawn through the point of clearest vision and the candle, in direction opposed to that of the candle's motion with respect to the point of clearest vision, and with a velocity equal to that of the candle.

4. The second method showed what is practically the same thing, namely, that the line of movement of the ghost was just as described; the direction the same as that of the point of clearest vision over the field in front; the velocity apparently about double that of the point of regard.

5. In both methods the ghost merged into the candle close to the point of direct vision, and in other positions seemed about equally removed from that point with the candle.